Math Background Assesment Test

Sample test with solutions

1. Factor the following polynomial into a product of linear terms: $x^2 - 4x + 3$.

2. Simplify the following expressions:
a) $(2x+3)^2 - x(2x-6) - (x+3)^2$;
b) $\left(\frac{2ab^2}{3xy}\right)^3 \left(\frac{4ax}{6y^3b^{-3}}\right)^{-2}$.

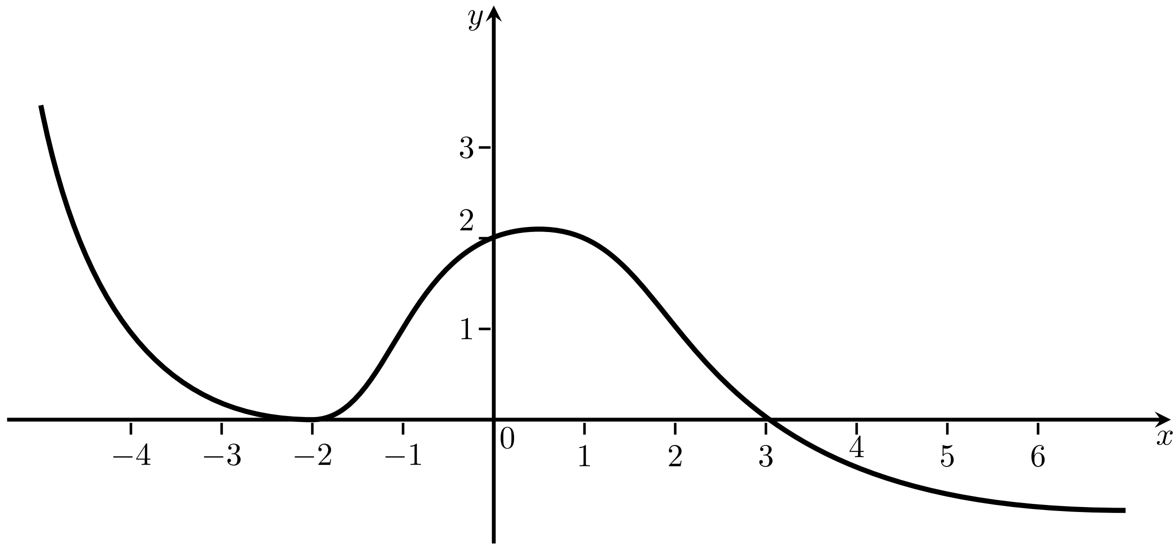
3. Add the following two fractions and then simplify: $\frac{y}{x-y} - \frac{2xy}{x^2-y^2}$.

4. Simplify as much as possible: $\frac{\sin(2x)}{\tan(x)} + \sin(\pi)$.

5. Solve the following equations:
a) $\frac{2e^x - 2}{x^2+1} = 0$;
b) $(x-2)\ln(x) = 0$
c) $\cos^2(x) - \sin^2(x) = 1$;
d) $|x-1| - 2x = 4$.

6. Solve the following inequalities, express the answer using intervals ("The set of all solutions is ... "):
a) $(x-2)(x^2+1) > 0$;
b) $\frac{x+1}{x-2} \geq 0$;
c) $x^2 - 4 < 0$;
d) $2x + |x-3| \geq 0$.

7. Consider the function f whose graph is in the following picture:



Based on this picture, answer the following questions:
a) What are the intercepts of the graph with the x axis?
b) What is the intercept of the graph and the y-axis?
c) What is $f(1)$?
d) Find all x satisfying $f(x) = 1$.

8. a) Draw a graph of the straight line connecting the points $(-1, 0)$ and $(2, 1)$.
b) Find the equation of this line.
c) Calculate the distance between the points $(-1, 0)$ and $(2, 1)$.

Solutions

1. $= (x-3)(x-1)$.

2. a) $= (4x^2+12x+9) - (2x^2-6x) - (x^2+6x+9) = x^2$.

b) $= \frac{8a^3b^6}{27x^3y^3} \cdot \frac{36y^6b^{-6}}{16a^2x^2} = \frac{8 \cdot 36a^3b^6y^6}{27 \cdot 16x^3y^3a^2x^2b^6} = \frac{2ay^3}{3x^5}$ pro $b, x, y \neq 0$.

3. $= \frac{y}{x-y} - \frac{2xy}{(x-y)(x+y)} = \frac{y(x+y)}{(x-y)(x+y)} - \frac{2xy}{(x-y)(x+y)} = \frac{yx+y^2-2xy}{(x-y)(x+y)}$

$= \frac{y^2-xy}{(x-y)(x+y)} = \frac{y(y-x)}{(x-y)(x+y)} = \frac{-y}{x+y}$.

4. $= \frac{2\sin(x)\cos(x)}{\frac{\sin(x)}{\cos(x)}} + 0 = 2\cos^2(x)$ pro $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

5. a) Multiplying both sides by x^2+1 (which is not zero) we obtain

$2e^x - 2 = 0 \implies e^x = 1 \implies x = 0$.

b) The expression is a product that is supposed to be zero, so one (or both) of the factors must be zero.

One possibility is that $x-2=0$. Then $x=2$. Since logarithm is defined there, we have a valid solution.

The second possibility is that $\ln(x) = 0$. Then necessarily $x = 1$.

Conclusions: The solutions are $x=1$ and $x=2$, or $x=1,2$ for short.

c) A least four ways.

Using $1 = \sin^2(x) + \cos^2(x)$ we get $\cos^2(x) - \sin^2(x) = \sin^2(x) + \cos^2(x)$

$\implies 2\sin^2(x) = 0 \implies \sin(x) = 0 \implies x = k\pi$ for $k \in \mathbb{Z}$.

Using the identity $\cos^2(x) - \sin^2(x) = \cos(2x)$ we get $\cos(2x) = 1 \implies 2x = 2k\pi$

$\implies x = k\pi$ for $k \in \mathbb{Z}$.

Using $1 = \sin^2(x) + \cos^2(x)$ we get $\sin^2(x) = 1 - \cos^2(x)$, substituting into the equation we obtain $\cos^2(x) - (1-\cos^2(x)) = 1 \implies \cos^2(x) = 1 \implies \cos(x) = \pm 1 \implies x = k\pi$ for $k \in \mathbb{Z}$.

One can also substitute $\cos^2(x) = 1 - \sin^2(x)$.

d) We need to remove the absolute value. Case 1: $(x-1) \geq 0$, that is, $x \geq 1$. Then $|x-1| = x-1$. So assuming $x \geq 1$ the equation becomes $x-1-2x = 4 \implies -x = 5 \implies x = -5$. Value $x=-5$ does not satisfy our assumption $x \geq 1$, so it is not a true solution.

Case 2: $(x-1) < 0$, that is, $x < 1$. Then $|x-1| = -(x-1)$. So assuming $x < 1$ the equation becomes $-(x-1) - 2x = 4 \implies -3x = 3 \implies x = -1$. Value $x = -1$ satisfies our assumption $x < 1$, so it is a true solution.

Conclusion: The only solution is $x = -1$.

6. a) The second term is always positive, so we can divide the inequality by it, obtaining $(x-2) > \frac{0}{x^2+1} \implies x-2 > 0 \implies x > 2$; the set of all solutions is $(2, \infty)$.

b) We see right away that the condition $x \neq 2$ is needed for the inequality to make sense.

Approach 1 (common sense): For a product of two numbers to be positive or zero, either both terms are positive (or zero) or both are negative (or zero). Thus there are two possibilities:

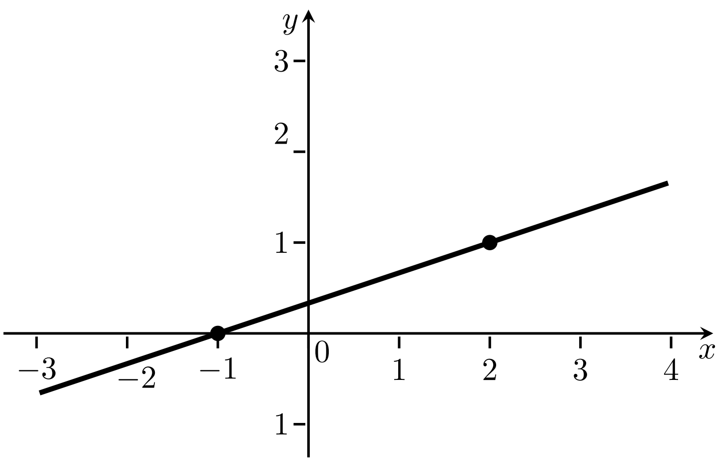
Case $x+1 \geq 0$ and $x-2 \geq 0$: This reads "$x \geq -1$ and $x \geq 2$", which determines the range $x \geq 2$ for solutions coming from this case.

Case $x+1 \leq 0$ and $x-2 \leq 0$: This reads "$x \leq -1$ and $x \leq 2$", which determines the range $x \leq -1$ for solutions coming from this case.

Since we had a choice which case to use, the two possibilities must be joined, that is, we use the operation union of sets. We recall that the case $x=2$ was forbidden, so the set of all solutions is $(-\infty, -1] \cup (2, \infty)$.

Approach 2 (algebra): We like to get rid of fractions, in this case we would like to multiply the inequality by $x-2$. However, we may be forced to change the direction of inequality

depending on the sign of the number we multiply with.
Case $x-2>0$: The inequality does not change after multiplying, we get $x+1\geq 0$, that is, $x\geq -1$. But this was done only under the assumption $x>2$, so numbers $x>2$ are solutions.
Case $x-2<0$: The inequality changes direction after multiplying, we get $x+1\leq 0$, that is, $x\leq -1$. This satisfies our assumption $x<2$, so the numbers $x\leq -1$ are all solutions.
Since we had a choice which case to use, we join the two possibilities, so the set of all solutions is $(-\infty,-1]\cup(2,\infty)$.
c) Polynomials in inequalities (and also equations) are best factored. Here we have $x^2-4=x^2-2^2=(x-2)(x+2)$. Thus the inequality reads $(x-2)(x+2)<0$.
Approach 1 (common sense): For a product of two numbers to be negative, one must be negative and the other positive. Thus there are two possibilities:
Case $x-2>0$ and $x+2<0$: This reads "$x>2$ and $x<-2$", which is not possible. So no contribution from this case.
Case $x-2<0$ and $x+2>0$: This reads "$x<2$ and $x>-2$", which determines the range $-2<x<2$ for solutions.
Conclusion: The set of all solutions is $(-2,2)$.
Approach 2 (algebra): We can simplify the inequality by dividing using one of the factors, for instance the factor $(x-2)$. However, we may be forced to change the direction of inequality depending on the sign of the number we divide with.
Case $x-2>0$: The inequality does not change after dividing, we get $x+2<0$, that is, $x<-2$. But this was done only under the assumption $x>2$, there are no numbers that would work.
Case $x-2<0$: The inequality changes direction after dividing, we get $x+2>0$, that is, $x>-2$. Combined with our assumption $x<2$ we see that numbers from the range $-2<x<2$ are solutions.
The two cases should be joined, so the set of all solutions is $(-2,2)$.
d) We remove the absolute value. Case 1: $(x-3)\geq 0$, then $|x-3|=x-3$. So assuming $x\geq 3$ the inequality becomes $2x+(x-3)\geq 0 \implies 3x\geq 3 \implies x\geq 1$. However, we were working under the assumption $x\geq 3$, so only values $x\geq 3$ are solutions for this case.
Case 2: $(x-3)<0$, then $|x-3|=-(x-3)$. So assuming $x<3$ the inequality becomes $2x-(x-3)\geq 0 \implies x\geq -3$. However, we were working under the assumption $x<3$, so only values $-3\leq x<3$ are solutions for this case.
The two cases should be joined, the set $[-3,3)\cup[3,\infty)$ can be written as one interval.
Conclusion: The set of all solutions is $[-3,\infty)$.
7. a) $x=-2$ and $x=3$. b) $(0,2)$ because $f(0)=2$. c) $f(1)=2$. d) $x\in\{-4,-1,2\}$.
8.



b) Slope: $k=\dfrac{1-0}{2-(-1)}=\frac{1}{3}$. Using the point $(0,-1)$ we get the equation $y-0=\frac{1}{3}(x-(-1))$, that is, $y=\frac{1}{3}(x+1)$. We can also write it as $3y=x+1$.
Using the point $(2,1)$ we would get $y-1=\frac{1}{3}(x-2)$, which again leads to the equation $3y=x+1$.

c) $d=\sqrt{(1-0)^2+(2-(-1))^2}=\sqrt{1^2+3^2}=\sqrt{10}$.

Math Background Assesment Test

Version 1

1. Factor the following polynomial into a product of linear terms: $x^2 - 2x - 3$.

2. Simplify the following expressions:
a) $x(x+4) - (x+2)^2$;
b) $\left(\frac{4a}{3b^3c}\right)^2 \left(\frac{2c^{-2}}{ab}\right)^{-4}$.

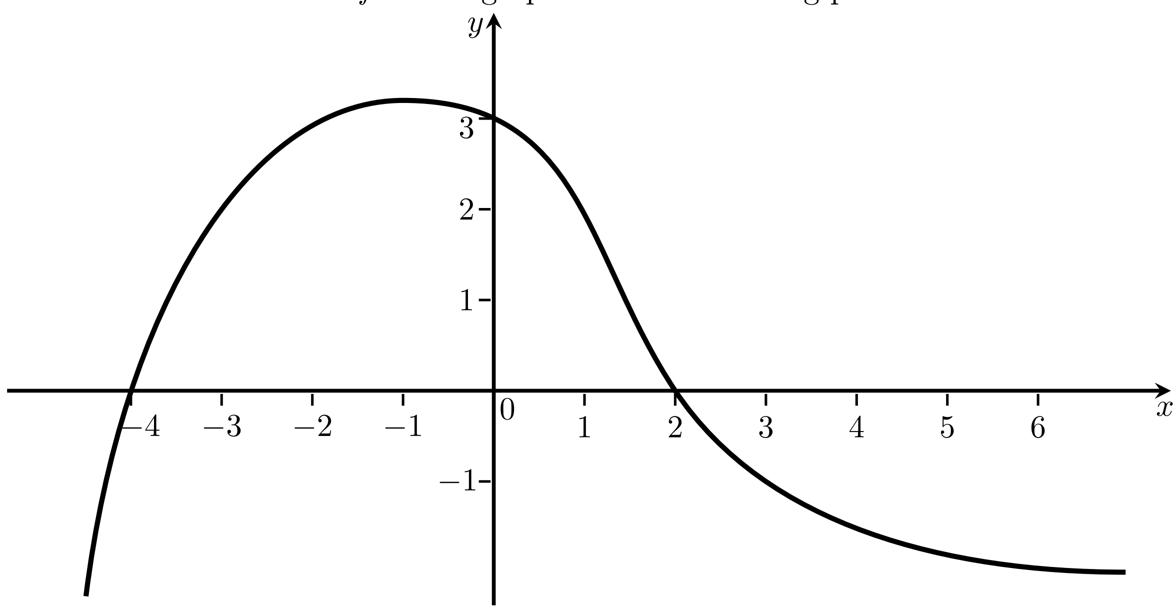
3. Add the following two fractions and then simplify: $\frac{x^2+y^2}{x^2-y^2} - \frac{y}{x-y}$.

4. Simplify as much as possible: $\cos(2x) - 2\sin^2(x) + \sin(0)$.

5. Solve the following equations:
a) $2e^{x^2-1} = 2$;
b) $\sin(2x) - 2\sin(x) = 0$;
c) $|x-2| + 2x = 1$.

6. Solve the following inequalities, express the answer using intervals ("The set of all solutions is ... "):
a) $\frac{x-1}{x^2+1} < 0$;
b) $\frac{x+2}{x-1} > 0$;
c) $2x - |x+2| \leq -1$.

7. Consider the function f whose graph is in the following picture:



Based on this picture, answer the following questions:
a) What are the intercepts of the graph with the x axis?
b) What is the intercept of the graph and the y-axis?
c) What is $f(3)$?
d) Find all x satisfying $f(x) = 2$.

8. a) Draw a graph of the straight line connecting the points $(-1,-1)$ and $(1,0)$.
b) Find the equation of this line.
c) Calculate the distance between the points $(-1,-1)$ and $(1,0)$.

Math Background Assesment Test
Version 2

1. Factor the following polynomial into a product of linear terms: $x^2 + x - 6$.

2. Simplify the following expressions:
a) $(2x-3)^2 - (x+3)^2$;
b) $\left(\frac{2x^3 z}{3^{-1}y^2}\right)^2 \left(\frac{2x^2 y}{z}\right)^{-3}$.

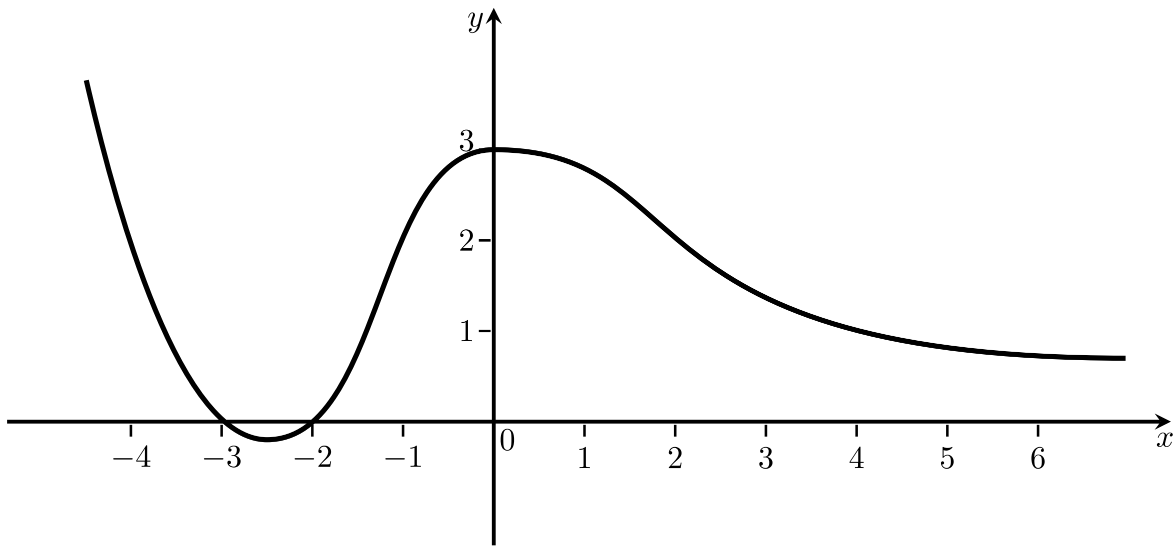
3. Add the following two fractions and then simplify: $\frac{x+y}{xy(x-y)} - \frac{x-y}{xy(x+y)}$.

4. Simplify as much as possible: $\tan(x)\sin(2x) - \cos(\pi)$.

5. Solve the following equations:
a) $\ln(x^2 - 3) = 0$;
b) $\tan(x) - \frac{1}{\cos(x)} = 0$;
c) $2x - |x-2| = 1$.

6. Solve the following inequalities, express the answer using intervals ("The set of all solutions is ... "):
a) $e^x(x-3) > 0$;
b) $(x-1)(x+3) \leq 0$;
c) $|x+3| + 2x < 6$.

7. Consider the function f whose graph is in the following picture:



Based on this picture, answer the following questions:
a) What are the intercepts of the graph with the x axis?
b) What is the intercept of the graph and the y-axis?
c) What is $f(4)$?
d) Find all x satisfying $f(x) = 2$.

8. a) Draw a graph of the straight line connecting the points $(-2, 2)$ and $(0, -2)$.
b) Find the equation of this line.
c) Calculate the distance between the points $(-2, 2)$ and $(0, -2)$.

Math Background Assesment Test

Version 3

1. Factor the following polynomial into a product of linear terms: $x^2 + 4x + 3$.

2. Simplify the following expressions:
a) $(2x+1)^2 - x(4x-3)$;
b) $\left(\frac{6u^2w^4}{v}\right)^3 \left(\frac{3w^3}{uv^2}\right)^{-4}$.

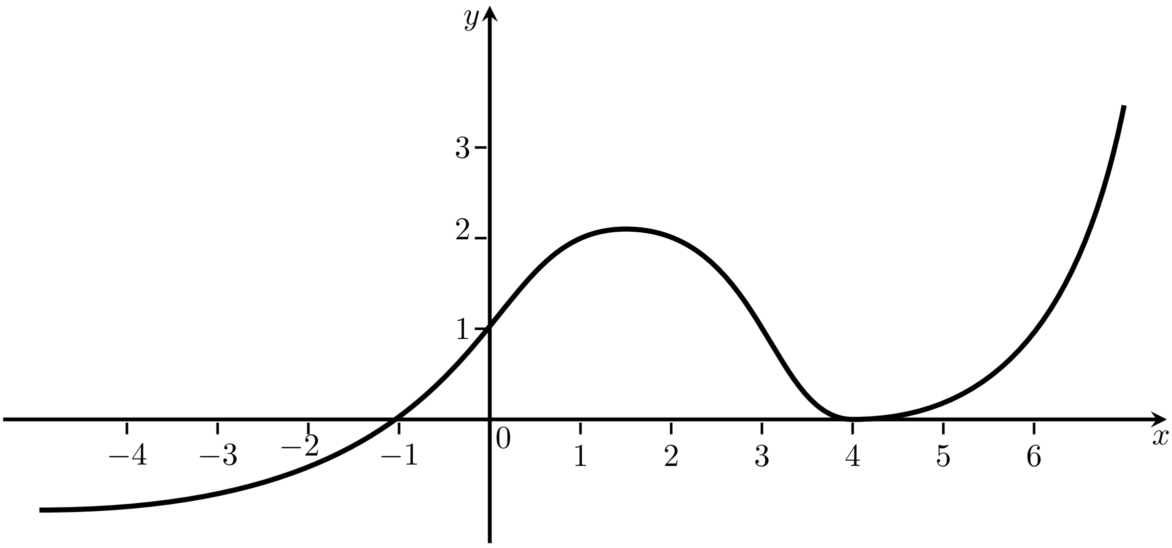
3. Add the following two fractions and then simplify: $\frac{y+1}{y(x-y)} - \frac{x+1}{xy(x+y)}$.

4. Simplify as much as possible: $\sin(2x) - \frac{2\sin^2(x)}{\tan(x)} + \sin(0)$.

5. Solve the following equations:
a) $x\, e^{x+1} = x$;
b) $\cos(2x) + \sin^2(x) = 0$;
c) $|x+1| - 2x = 5$.

6. Solve the following inequalities, express the answer using intervals ("The set of all solutions is ... "):
a) $x^4(x+3) \geq 0$;
b) $\frac{x-3}{x-2} < 0$;
c) $3x + |x-2| > 2$.

7. Consider the function f whose graph is in the following picture:



Based on this picture, answer the following questions:
a) What are the intercepts of the graph with the x axis?
b) What is the intercept of the graph and the y-axis?
c) What is $f(1)$?
d) Find all x satisfying $f(x) = 1$.

8. a) Draw a graph of the straight line connecting the points $(0, -1)$ and $(3, 5)$.
b) Find the equation of this line.
c) Calculate the distance between the points $(0, -1)$ and $(3, 5)$.

Math Background Assesment Test

Version 4

1. Factor the following polynomial into a product of linear terms: $x^2 - 6x + 8$.

2. Simplify the following expressions:
a) $(x+2)(2x+6) - (x+3)^2$;
b) $\left(\frac{4r^2}{s^3t^{-2}}\right)^{-2}\left(\frac{2r}{s^2t^3}\right)^3$.

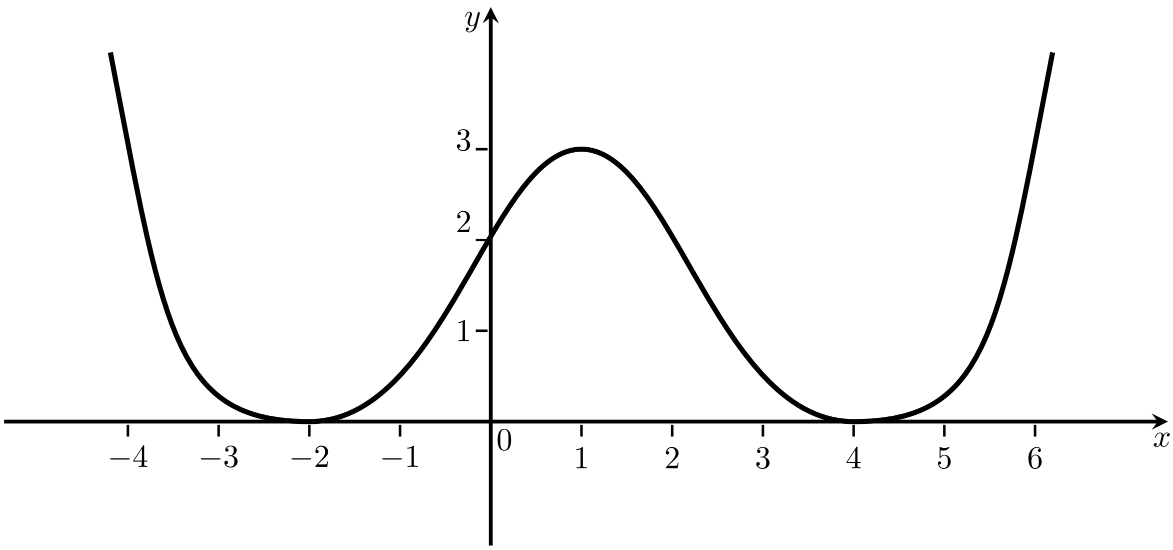
3. Add the following two fractions and then simplify: $\frac{1}{x-y} - \frac{2y}{x^2-y^2}$.

4. Simplify as much as possible: $\frac{\cos(0)}{\cos^2(x)} - \tan^2(x)$.

5. Solve the following equations:
a) $\frac{\ln(x-1)}{x^2+1} = 0$;
b) $\frac{\sin(2x)}{\cos^2(x)} - \tan(x) = 0$;
c) $3x + |x+2| = 2$.

6. Solve the following inequalities, express the answer using intervals ("The set of all solutions is ... "):
a) $\frac{x-2}{x^4} > 0$;
b) $(x-2)(x+3) \geq 0$;
c) $|x+1| - 2x \geq 1$.

7. Consider the function f whose graph is in the following picture:



Based on this picture, answer the following questions:
a) What are the intercepts of the graph with the x axis?
b) What is the intercept of the graph and the y-axis?
c) What is $f(2)$?
d) Find all x satisfying $f(x) = 3$.

8. a) Draw a graph of the straight line connecting the points $(-1, 1)$ and $(3, -1)$.
b) Find the equation of this line.
c) Calculate the distance between the points $(-1, 1)$ and $(3, -1)$.